MATH6031 Lecture 2

Last time : $\mathfrak{g}$ Lie algebra / k, $\mathrm{char}(k)=0$

$\rightsquigarrow$ • $S(\mathfrak{g})$: symmetric algebra

$$S(\mathfrak{g}) = T(\mathfrak{g}) / (x\otimes y - y\otimes x,\ x,y\in\mathfrak{g})$$

• $U(\mathfrak{g})$: universal enveloping algebra

$$U(\mathfrak{g}) = T(\mathfrak{g}) / (x\otimes y - y\otimes x - [x,y],\ x,y\in\mathfrak{g})$$

<u>Thm</u> (PBW) The symmetrization map

$$I_{PBW} : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$$
$$x_1 \cdots x_n \longmapsto \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} x_{\sigma_1}\cdots x_{\sigma_n}$$

is an isom of filtered vector spaces.

But I_{PBW} is <u>not</u> an algebra homomorphism.

Duflo elt $J \in \hat{S}^{\cdot}(\mathfrak{g}^*)$

$$J(x) := \det\left(\frac{1-e^{-ad_x}}{ad_x}\right)$$

<u>Thm</u> (Duflo) The composition

$$S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{J^{\frac{1}{2}}} S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{I_{PBW}} U(\mathfrak{g})^{\mathfrak{g}} = Z(U(\mathfrak{g}))$$

is an isomorphism of algebras

§ <u>Chevalley - Eilenberg cohomology</u>

"An intro. to homological algebra" by C. Weibel.

V : $\mathfrak{g}$ - module

<u>Def</u> The Chevalley-Eilenberg complex associated to V is defined as $C^\cdot(\mathfrak{g}, V)$,

$$C^n(\mathfrak{g},V) = (\Lambda^n \mathfrak{g})^* \otimes V$$
$$= \{\text{linear maps } \Lambda^n \mathfrak{g} \to V\}$$

and

$$(d_c(\ell))(x_0,\dots,x_n)$$
$$:= \sum_{0 \le i < j \le n} (-1)^{i+j} \ell([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n)$$
$$+ \sum_{i=0}^{n} (-1)^i x_i \cdot \ell(x_0, \dots, \hat{x}_i, \dots, x_n)$$

We have $d_c^2 = 0$. So we have the Chevalley-Eilenberg cohomology $H^\cdot(\mathfrak{g}, V)$.

- $H^0(\mathfrak{g}, V) = V^{\mathfrak{g}}$ = space of $\mathfrak{g}$-invariant elts in V
- For $H^1(\mathfrak{g}, V)$,

$$\text{1-cocycles} = \{\ell : \mathfrak{g} \to V : \ell([x,y]) = x \cdot \ell(y) - y \cdot \ell(x) \ \forall x, y \in \mathfrak{g}\}$$
$$= \{V\text{-valued derivations on } \mathfrak{g}\}$$

$$\text{1-coboundaries} = \left\{ \ell_v : \begin{array}{l} \mathfrak{g} \to V \\ x \mapsto x \cdot v \end{array} \right\} = \{\text{inner derivations}\}$$

this is called an inner derivation.

$$H^1(\mathfrak{g}, V) = \frac{\{\mathfrak{g}\text{-derivations}\}}{\{\text{inner derivations}\}} = \{\text{equiv. classes of } \mathfrak{g}\text{-module ext}^{\text{ns}} \text{ of } k \text{ by } V\}$$

- For $H^2(\mathfrak{g}, V)$,

2-cocycles are linear maps

$$\omega : \Lambda^2 \mathfrak{g} \to V$$

s.t. $\omega([x,y],z) + \omega([z,x],y) + \omega([y,z],x)$

$$- x\cdot\omega(y,z) + y\cdot\omega(x,z) - z\cdot\omega(y,z) = 0$$

$\forall x,y,z \in \mathfrak{g}$

$\Longleftrightarrow$ $\mathfrak{g}\oplus V$ with the bracket

$$[(x,u),(y,v)] = ([x,y],\, x\cdot v - y\cdot u + \omega(x,y))$$

is a Lie algebra

This is called an extension of $\mathfrak{g}$ by V.

2-coboundaries $\omega = d_c(\ell)$ correspond to trivial extensions, i.e. the Lie algebra structure on $\mathfrak{g}\oplus V$ is isomorphic to the trivial one given by $\omega_0 = 0$.

$\leadsto$ $H^2(\mathfrak{g}, V) = \{$equiv. classes of Lie algebra ext$^{\underline{ns}}$ of $\mathfrak{g}$ by $V\}$

Now, back to Duflo.

as $\mathfrak{g}$-modules

$$I_{PBW} : S(\mathfrak{g}) \xrightarrow{\cong} U(\mathfrak{g})$$

$$\leadsto I_{PBW} : C^\cdot(\mathfrak{g}, S(\mathfrak{g})) \xrightarrow{\cong} C^\cdot(\mathfrak{g}, U(\mathfrak{g}))$$

<u>Thm</u> (extended Duflo isom.)

$I_{PBW} \circ J^{\frac{1}{2}}$ induces an isomorphism of algebras at the level of cohomology

$$\boxed{H^\cdot(\mathfrak{g}, S(\mathfrak{g}))} \xrightarrow{\cong} \boxed{H^\cdot(\mathfrak{g}, U(\mathfrak{g}))}$$

The deg 0 part of this thm recovers the original

Duflo isom:

$$H^0(\mathfrak{g}, S(\mathfrak{g})) = S(\mathfrak{g})^{\mathfrak{g}}$$

$$H^0(\mathfrak{g}, U(\mathfrak{g})) = U(\mathfrak{g})^{\mathfrak{g}}$$

§ Hochschild cohomology

A : associative algebra

M : A-bimodule

Def The Hochschild complex $C^\cdot(A, M)$ is defined by

$$C^n(A,M) = \{ f : A^{\otimes n} \longrightarrow M \text{ linear maps} \}$$

w/ differential d_H

$$(d_H(f))(a_0, \dots, a_n) := a_0 \cdot f(a_1, \dots, a_n) + \sum_{i=1}^{n} (-1)^i f(a_0, \dots, a_{i-1}a_i, \dots, a_n) + (-1)^{n+1} f(a_0, \dots, a_{n-1}) \cdot a_n$$

We have $d_H^2 = 0 \rightsquigarrow$ Hochschild cohomology

$$H^\cdot(A,M) = H^\cdot(C^\cdot(A,M), d_H)$$

Rmk: If $M = B$ is an algebra s.t. $\forall a \in A$ and $b, b' \in B$,

$$a(bb') = (ab)b' \quad \text{and} \quad (bb')a = b(b'a)$$

then $(C^\cdot(A,B), d_H)$ is a differential graded algebra (DGA)

with the product $\cup$ defined by

$$(f \cup g)(a_1, \dots, a_{m+n}) = f(a_1, \dots, a_n)\, g(a_{n+1}, \dots, a_{m+n})$$

In particular, we can take $M = A$ and we will write
$$HH^\bullet(A) := H^\bullet(A, A).$$

- $H^0(A, M) = \{ m \in M : a \cdot m = m \cdot a \ \forall a \in A \}$
$= $ space of A-invariant elts in M
$=: M^A$

e.g. $HH^0(A) = Z(A)$.

- For $H^1(A, M)$,

1-cocycles are linear maps $\ell : A \to M$
s.t. $\ell(ab) = a\ell(b) + \ell(a) \cdot b \quad \forall a, b \in A$

$\Longleftrightarrow$ A-derivations with values in M

1-coboundaries are derivations of the form
$$\ell_m : A \to M \qquad a \mapsto m \cdot a - a \cdot m \qquad \text{for some } m \in M$$
which are called inner derivations

$$\rightsquigarrow \quad H^1(A, M) = \frac{\{A\text{-derivations}\}}{\{\text{inner derivations}\}}$$

$k[\varepsilon]/\varepsilon^2$

Now restrict ourselves to the case $M = A$.

Def An infinitesimal deformation of A is an associative ε-linear product $*$ on $A[\varepsilon]/\varepsilon^2$
s.t. $a * b \equiv ab \pmod{\varepsilon}$

i.e. $a * b = ab + \mu(a, b)\varepsilon$
for some map $\mu : A \otimes A \to A$. $\in C^2(A, A)$

$*$ is assoc. $\Longleftrightarrow$ $a\mu(b, c) + \mu(a, bc) = \mu(a, b)c + \mu(ab, c)$

$*$ is assoc. $\iff a\mu(b,c) + \mu(a,bc) = \mu(a,b)c + \mu(ab,c)$

$\iff \mu$ is a 2-cocycle

Furthermore, two infinitesimal deformations $*$ and $*'$ are equivalent if $\exists$ an isomorphism Φ of $k[\varepsilon]/\varepsilon^2$-algebras

$$\downarrow$$

$$= A$$

$\iff \exists\, \ell : A \to A$ s.t. the isom Φ maps $a \mapsto a + \ell(a)\varepsilon$

Note that Φ is a morphism

$\iff \mu(a,b) + \ell(ab) = \mu'(a,b) + a\cdot\ell(b) + \ell(a)\cdot b$

$\iff \mu - \mu' = d_H(\ell)$

In conclusion, $HH^2(A) = $ equiv. classes of infinitesimal deform$^{\underline{\text{ns}}}$ of A

- An order n $(n>0)$ deformation of A is an associative ε-linear product $*$ on the $k[\varepsilon]/\varepsilon^{n+1}$-algebra $A[\varepsilon]/\varepsilon^{n+1}$ s.t. $a*b \equiv ab \pmod{\varepsilon}$

$$\iff a*b = ab + \sum_{i=1}^{n} \mu_i(a,b)\,\varepsilon^i$$

for some bilinear maps $\mu_i : A\otimes A \to A$.

Setting $$\mu := \sum_{i=1}^{n} \mu_i\,\varepsilon^i \in C^2(A, A[\varepsilon]).$$

Then

$*$ is associative $\iff d_H(\mu)(a,b,c)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\ \|\!|$

$$\mu(\mu(a,b),c)-\mu(a,\mu(b,c)) \pmod{\varepsilon^{n+1}}$$

Prop (Gerstenhaber)

If $*$ is an order n deformation, then the linear map

$$\nu_{n+1} : A^{\otimes 3} \longrightarrow A$$

defined by

$$\nu_{n+1}(a,b,c) := \sum_{i=1}^{n} \mu_i(\mu_{n+1-i}(a,b),c)-\mu_i(a,\mu_{n+1-i}(b,c))$$

is a 3-cocycle, i.e. $d_H(\nu_{n+1})=0$

Pf: • set $\nu(a,b,c) := \mu(\mu(a,b),c)-\mu(a,\mu(b,c)) \in A[\varepsilon]$.

• Then assoc. of $* \iff \boxed{d_H(\mu)\equiv\nu} \pmod{\varepsilon^{n+1}}$

Want to show that $d_H(\nu)\equiv 0 \pmod{\varepsilon^{n+2}}$

Now
$$\begin{aligned} d_H(\nu)(a,b,c,d) &= \mu(a,d_H(\mu)(b,c,d))-d_H(\mu)(\mu(a,b),c,d) \\ &\quad + d_H(\mu)(a,\mu(b,c),d) - d_H(\mu)(a,b,\mu(c,d)) \\ &\quad + \mu(d_H(\mu)(a,b,c),d) \\ &\overset{(\mathrm{mod}\ \varepsilon^{n+2})}{\equiv} \mu(a,\nu(b,c,d))-\nu(\mu(a,b),c,d) \\ &\quad + \nu(a,\mu(b,c),d) - \nu(a,b,\mu(c,d)) \\ &\quad + \mu(\nu(a,b,c),d) \\ &= 0 \end{aligned}$$

Given an order n deformation, $d_H(\nu_{n+1})=0$

$$\Rightarrow \underline{[\nu_{n+1}]} \in HH^3(A)$$

Extending it to an order $(n+1)$ deformation

$\iff \exists \mu_{n+1} : A \otimes A \to A$

s.t. $\displaystyle\sum_{i=0}^{n+1} \mu_i(\mu_{n+1-i}(a,b),c) = \sum_{i=0}^{n+1} \mu_i(a,\mu_{n+1-i}(b,c))$

$\iff \quad d_H(\mu_{n+1}) = \nu_{n+1}$

$\iff \quad [\nu_{n+1}] = 0 \in HH^3(A)$

This gives a deformation-obstruction theory in terms of $HH^\cdot(A)$.

§ Chevalley-Eilenberg vs Hochschild cohomology

M : $U(\mathfrak{g})$-bimodule $\longrightarrow$ $H^\cdot(U(\mathfrak{g}), M)$ Hochschild cohomology

Equip M with a $\mathfrak{g}$-module structure via $\to H^\cdot(\mathfrak{g}, M)$ Chevalley-Eilenberg cohomology

$x \cdot m = x \cdot m - m \cdot x$ for $x \in \mathfrak{g}$ and $m \in M$.

Thm (1) There is an isom of graded vector spaces

$$H^\cdot(\mathfrak{g}, M) \cong H^\cdot(U(\mathfrak{g}), M)$$

(2) If $M = A$ is equipped with a $U(\mathfrak{g})$-invariant associative product, then the above isom. is is an isom. of graded algebras

We'll only look at the case $M = U(\mathfrak{g})$, (i.e.. $HH^\cdot(U(\mathfrak{g}))$ on the RHS).

Sketch of Pf: First we consider the composition

$$\begin{tikzcd} \Lambda^n \mathfrak{g} \arrow[r, hook] \arrow[rr, bend left, "\varepsilon"] & \Lambda^n U(\mathfrak{g}) \arrow[r] & U(\mathfrak{g})^{\otimes n} \end{tikzcd}$$

inclusion   antisymmetrization map

$$a_1 \wedge \cdots \wedge a_n \mapsto \sum_{\sigma \in \mathfrak{S}_n} (-1)^{|\sigma|} a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}$$

Pulling back gives

$$C^n(U(\mathfrak{g}), U(\mathfrak{g})) \xrightarrow{\varepsilon^*} \mathrm{Hom}(\Lambda^n \mathfrak{g}, U(\mathfrak{g})) = (\Lambda^n \mathfrak{g})^* \otimes U(\mathfrak{g})$$

- Then the following diagram commutes

$$\begin{tikzcd} C^n(U(\mathfrak{g}), U(\mathfrak{g})) \arrow[r, "\varepsilon^*"] \arrow[d, "d_H"] \arrow[dr, phantom, "\circlearrowright"] & \Lambda^n \mathfrak{g}^* \otimes U(\mathfrak{g}) \arrow[d, "d_C"] \\ C^{n+1}(U(\mathfrak{g}), U(\mathfrak{g})) \arrow[r] & \Lambda^{n+1} \mathfrak{g}^* \otimes U(\mathfrak{g}) \end{tikzcd}$$

- Note that the filtration $U(\mathfrak{g})$ induces a filtration on $C^\cdot(U(\mathfrak{g}), U(\mathfrak{g}))$ by setting

$$F^p C^n(U(\mathfrak{g}), U(\mathfrak{g})) = \{ f : F^{i_1} U(\mathfrak{g}) \otimes \cdots \otimes F^{i_n} U(\mathfrak{g}) \to F^{n-i_0} U(\mathfrak{g}) : i_0 + i_1 + \cdots + i_n = p \}$$

and a filtration on $\Lambda^\cdot \mathfrak{g}^* \otimes \underline{U(\mathfrak{g})}$ by setting

$$F^p(\Lambda^n \mathfrak{g}^* \otimes U(\mathfrak{g})) = \{ \ell : \Lambda^n \mathfrak{g} \to F^{n-p} U(\mathfrak{g}) \}$$

- Key: ε^* respects these filtrations and hence the assoc. spectral sequences
- So it suffices to compare the E_1-page
  - By the PBW thm, the E_1-page of the RHS is $\Lambda^n \mathfrak{g}^* \otimes S^{n-p}(\mathfrak{g})$
  - Fact: $\Lambda^n \mathfrak{g}^* \otimes S^\cdot(\mathfrak{g}) \cong \underbrace{HH^n(S^\cdot(\mathfrak{g}))}_{*}$

(Koszul

- Fact : $\Lambda^{\cdot}\mathfrak{g}^{*} \otimes S^{\cdot}(\mathfrak{g}) \cong \underbrace{HH^{\cdot}(S^{\cdot}(\mathfrak{g}))}_{\Uparrow}$
(Koszul duality)

E_1-page of the LHS

$\to$ We have isom on the E_1-page

$\Rightarrow$ Isom. on the abutment

i.e. $HH^{\cdot}(U(\mathfrak{g})) \cong H^{\cdot}(\mathfrak{g}, U(\mathfrak{g}))$. #

Therefore, we can rewrite the extended Duflo isom. as

$$H^{\cdot}(\mathfrak{g}, S(\mathfrak{g})) \xrightarrow{\cong} HH^{\cdot}(U(\mathfrak{g}))$$

- Chevalley-Eilenberg Lie algebra cohomology
- deformations of the (canonical) Poisson bracket on $\underline{\underline{\mathfrak{g}^*}}$

- Hochschild cohomology
- deformations of the assoc. product on $U(\mathfrak{g})$
- deformations of star products on $\underline{\mathfrak{g}^*}$